Inverse Perturbation Method for Inverse Eigenvalue Problem Based on Finite Element Analysis

MIAO Yu-bin (苗玉彬), LIU Cheng-liang (刘成良), CAO Qi-xin (曹其新)
School of Mechanical Engineering, Shanghai Jiaotong University, Shanghai, 200051
Jay Lee (李杰)
The University of Wisconsin, Milwaukee, USA

An inverse perturbation method is described for solving the general inverse eigenvalue problem. By taking the analysis of the rotor system as example based upon FEM, the new inverse perturbation method for structural design with specified low-order natural frequencies or frequency constraint bands is detailed as well as its complete theoretical basis. Moreover, formulations to calculate the inverse perturbation parameter ε and method to select the corresponding ε's value properly are also proposed. The proposed method is characterized in reducing frequency analysis and suitable for large and small structural changes alike. Finally, several different numerical examples for inverse eigenvalue problem are discussed to illustrate the method, which show that this inverse perturbation method is general and can be applied to other type of structure or element.

Keywords: *Inverse Eigenvalue Problem, Perturbation, Inverse Perturbation Method*

Introduction

The design of a structure either with specified natural frequencies or with specified frequency constraint bands is a problem that often appears in mechanical or structural engineering. The most basic feature in determining the vibration behavior of a structure is its eigenvalues and eigenvectors associated with each natural frequency. In most cases the first design does not satisfy all the free vibration objectives and/or practical constraints. Therefore, it's important for the designer to ensure that the natural frequencies of the structure do not coincide with the excitation frequencies. The common industrial practice for optimizing the vibration behavior of structures is to conduct a series of modifications, which is usually only slightly different from the previous structure. Usually it implies a number of expensive finite element analysis (FEA) design iterations.

To eliminate the need to re-analyze the whole structure, research efforts were conducted towards developing new concepts with sufficient information to find the exact modified parameters, which yield the required natural frequencies. Early work in this direction[1,2] utilized the first order terms of a Taylor's series expansion and is based on Rayleigh's work. Chen and Garba[3] used the iterative method to modify structural systems. Later Baldwin and Hutton[4] presented a detailed review of structural modification techniques.

The inverse eigenvalue problem is to determine the set of parameters so that the associated structure has specified eigenvalues. And the numerical definition of the inverse eigenvalue problem can be given as following:

Suppose $A(x)$ is derived from the equation

$$A(x) = A_0 + \sum_{i=1}^{n} x_i A_i \tag{1}$$

where $x \in \boldsymbol{R}^n$, $\{A_i\}$ are real symmetric n matrices. Let $\{\lambda_i(x)\}$ be the eigenvalues of $A\{x\}$ arranged in ascending order: $\lambda_1(x) \leqslant \cdots \leqslant \lambda_n(x)$.

The eigenvalue problem is generally given as follows:

$$K\varphi_j = \lambda_j M\varphi_j,\ j = 1,\ 2,\ \cdots,\ m \tag{2}$$

where K is the stiffness matrix; M is the mass matrix; φ_j and λ_j are defined as jth eigenvector and eigenvalue of the system, m is the number of degree of freedom.

Therefore, the inverse eigenvalue problem can be defined as

Given real numbers $\lambda_1^* \leqslant \cdots \leqslant \lambda_n^*$, find $x \in R^n$, so that $\lambda_i(x) = \lambda_i^*$, $i = 1, 2, \cdots, n$.

Three kinds of methods are being developed and used in solving the inverse eigenvalue problem:

A. Mathematical optimization method: Let k_{pq} and m_{pq} be the function of design variable X, K, M be the fonctionelle of X, λ^* be the objective function, then Eq. (2) can be solved as a mathematical problem using mathematical optimization approach, for example, Lagrange multiplier method, gradient-type subspace iterative method and feasible direction method[5-8].

B. Matrix perturbation method: the solution of Eq. (2) can be written as analytic function of ε as follows according to the theory of perturbation[9,10]:

$$\left.\begin{aligned} \varphi_j &= \varphi_{j0} + \varepsilon\varphi_{j1} + \varepsilon^2\varphi_{j2} + \cdots \\ \lambda_j &= \lambda_{j0} + \varepsilon\lambda_{j1} + \varepsilon^2\lambda_{j2} + \cdots \end{aligned}\right\} \tag{3}$$

Received Dec. 9, 2002
This research is supported by China National Natural Science Foundation (CNNSF), Research Grant No. 50128504

Once known the prescribed eigenvalue λ_{j0} and the corresponding mode shape φ_{j0}, the new eigenvalue λ_j can be computed expediently from Eq. (3).

C. Sensitivity analysis method: Sensitivity analysis is used to assess the effect of varying structural parameters on the dynamic characteristic of structure. This can help us not only to determine which structural parameters should be perturbed but also to find the perturbation direction[11,12].

However, method A needs repeated iterations of frequency analysis, which usually means time consuming and expensive FEA, and it is often not very successful since it does not have advantage of solving non-linear problem. Method B may reduce frequency analysis to some extent, but the perturbation parameter ε is hard to set; and though method C can find the perturbation direction, it can't give the corresponding perturbative amplitude of the design variables.

Thus it is easy to think of finding an inverse eigenvalue solution method when λ_j^* given to avoid repeated iterations of frequency analysis and to set the perturbation parameter ε more accurately.

Ref.13 proposed the idea of computing design variables $\boldsymbol{X}$ from the given eigenvalue λ^* according to the theory of general inverse eigenvalue problem. Ref.14 improved the method to solve the finite element optimization design problem of rotor system with frequency constraints. But it seems that their theoretical basis is not complete and they have not proposed the general equation of the perturbation parameter ε as well. The fundamental difference between method A and the method described in this paper is that instead of using iterative optimization method, the perturbation parameter ε is directly computed with an analytic equation derived from the dynamic analysis of finite-element models. The method is very efficient because it doesn't need repeat analysis. Compared with method B and C, our method can give the estimated value of the perturbation parameter ε. Based on this we describe the improvement of this method, which make its theoretical basis more complete. An example of beam element of rotor system is also illustrated to propose the calculation equation and accurate and simplified evaluation method of the perturbation parameter ε.

Theoretical Basis

The frequency gradient equation between the eigenvalue can be obtained by differentiating Eq. (2)

$$\partial\lambda_j/\partial x_i = \varphi_j^T \partial\boldsymbol{K}/\partial x_i \varphi_j - \lambda_j \varphi_j^T \partial\boldsymbol{M}/\partial x_i \varphi_j \tag{4}$$

Approximately we have

$$\Delta\lambda_j = \varphi_j^T \Delta\boldsymbol{K}\varphi_j - \lambda_j \varphi_j^T \Delta\boldsymbol{M}\varphi_j \tag{5}$$

where $\Delta\lambda_j$, $\Delta\boldsymbol{K}$, $\Delta\boldsymbol{M}$ are the increment of λ_j, $\boldsymbol{K}$, $\boldsymbol{M}$ respectively. And j is the exponent number of eigenvalue. x_i is the ith design variable, $i = 1, 2, \cdots, n$.

As it illustrated above, the function relation of design variable $\boldsymbol{X}$ or ε and $\boldsymbol{K}$, $\boldsymbol{M}$ can be obtained from the finite element analysis mode. An example of finite element analysis mode of the plane beam element of rotor system whose design variables are $\boldsymbol{X}_s$, $\boldsymbol{X}_l$, $\boldsymbol{X}_h$ is detailed as following to further illustrate it.

The element stiff matrix can be decomposed as

$$\boldsymbol{K}^e = EI\begin{bmatrix} 12/\boldsymbol{X}_l^3 & & & \\ 6/\boldsymbol{X}_l^2 & 4/\boldsymbol{X}_l & & \\ -12/\boldsymbol{X}_l^3 & -6/\boldsymbol{X}_l^2 & 12/\boldsymbol{X}_l^3 & \\ 6/\boldsymbol{X}_l^2 & 2/\boldsymbol{X}_l & -6/\boldsymbol{X}_l^2 & 4/\boldsymbol{X}_l \end{bmatrix} =$$

$$EI\begin{bmatrix} 12/\boldsymbol{X}_l^3 & & & \\ 0 & 0 & & \\ -12/\boldsymbol{X}_l^3 & 0 & 12/\boldsymbol{X}_l^3 & \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$EI\begin{bmatrix} 0 & & & \\ 6/\boldsymbol{X}_l^2 & 0 & & \\ 0 & -6/\boldsymbol{X}_l^2 & 0 & \\ 6/\boldsymbol{X}_l^2 & 0 & -6/\boldsymbol{X}_l^2 & 0 \end{bmatrix} +$$

$$E\cdot I\begin{bmatrix} 0 & & & \\ 0 & 4/\boldsymbol{X}_l & & \\ 0 & 0 & 0 & \\ 0 & 2/\boldsymbol{X}_l & 0 & 4/\boldsymbol{X}_l \end{bmatrix} =$$

$$\boldsymbol{X}_s^2(\overline{\boldsymbol{K}}_1^e/\boldsymbol{X}_l^3 + \overline{\boldsymbol{K}}_2^e/\boldsymbol{X}_l^2 + \overline{\boldsymbol{K}}_3^e/\boldsymbol{X}_l) \tag{6}$$

where $\boldsymbol{K}^e$ is the element stiff matrix; $\overline{\boldsymbol{K}}_1^e$, $\overline{\boldsymbol{K}}_2^e$, $\overline{\boldsymbol{K}}_3^e$ are constant matrixes.

The bearing support rigidity parameter $\boldsymbol{X}_h$ of rotor system should be added into matrix element of stiff matrix with respect to the influence of supporting constraint condition. From Eq. (6), the element matrices constitute the whole stiff matrix

$$\boldsymbol{K} = \boldsymbol{X}_s^2(\overline{\boldsymbol{K}}_1/\boldsymbol{X}_l^3 + \overline{\boldsymbol{K}}_2/\boldsymbol{X}_l^2 + \overline{\boldsymbol{K}}_3/\boldsymbol{X}_l) + \boldsymbol{X}_h\overline{\boldsymbol{K}}_h \tag{7}$$

where $\overline{\boldsymbol{K}}_1$, $\overline{\boldsymbol{K}}_2$, $\overline{\boldsymbol{K}}_3$ are stiffness matrices of structure constant; $\boldsymbol{X}_h$ is the coefficient of supporting rigidity.

Considering the iterate number k and exponent number j of the eigenvalue, let

$$\left.\begin{aligned} \boldsymbol{X}_j^{(k+1)} &= (1+\varepsilon_j^{(k+1)})\boldsymbol{X}_j^{(k)} \\ \Delta\boldsymbol{K}_j^{(k+1)} &= \boldsymbol{K}_j^{(k+1)} - \boldsymbol{K}_j^{(k)} \\ \Delta\boldsymbol{M}_j^{(k+1)} &= \boldsymbol{M}_j^{(k+1)} - \boldsymbol{M}_j^{(k)} \end{aligned}\right\} \tag{8}$$

Suppose all of the design variables $\boldsymbol{X}$ are perturbed by the same perturbation parameter ε, then

$$\Delta\boldsymbol{K}_j^{(k+1)} = \boldsymbol{K}_j^{(k+1)} - \boldsymbol{K}_j^{(k)} = -\varepsilon_j^{(k+1)}\frac{(\boldsymbol{X}_{sj}^{(k)})^2\overline{\boldsymbol{K}}_1^{(k)}}{(\boldsymbol{X}_{lj}^{(k)})^3(1+\varepsilon_j^{(k+1)})} + \frac{\varepsilon_j^{(k+1)}(\boldsymbol{X}_{sj}^{(k)})^2\overline{\boldsymbol{K}}_3^{(k)}}{\boldsymbol{X}_{lj}^{(k)}} + \varepsilon_j^{(k+1)}\boldsymbol{X}_{hj}^{(k)}\overline{\boldsymbol{K}}_h^{(k)} \tag{9a}$$

If we expand the binomial $\frac{1}{1+\varepsilon_j^{(k+1)}}$, then

$$\Delta\boldsymbol{K}_j^{(k+1)} = \varepsilon_j^{(k+1)}\boldsymbol{X}_{hj}^{(k)}\overline{\boldsymbol{K}}_h^{(k)} - \varepsilon_j^{(k+1)}[1-\varepsilon_j^{(k+1)} + (\varepsilon_j^{(k+1)})^2 - (\varepsilon_j^{(k+1)})^3 + \cdots]\frac{(\boldsymbol{X}_{sj}^{(k)})^2\overline{\boldsymbol{K}}_1^{(k)}}{(\boldsymbol{X}_{lj}^{(k)})^3} +$$

$$\frac{\varepsilon_j^{(k+1)}(X_{ij}^{(k)})^2\overline{K}_3^{(k)}}{X_{ij}^{(k)}} \tag{9b}$$

In the same argument the whole mass matrix can be obtained as

$$M = X_i(X_i\overline{M}_1 + X_i^2\overline{M}_2 + X_i^3\overline{M}_3) \tag{10}$$

And the whole incremental mass matrix is

$$\begin{aligned}\Delta M^{(k+1)} = {} & [(\varepsilon_j^{(k+1)})^2 + 2\varepsilon_j^{(k+1)}]X_{ij}^{(k)}X_{ij}^{(k)}\overline{M}_1^{(k)} + \\ & [(\varepsilon_j^{(k+1)})^3 + 3(\varepsilon_j^{(k+1)})^2 + 3\varepsilon_j^{(k+1)}] \cdot \\ & X_{ij}^{(k)}(X_{ij}^{(k)})^2\overline{M}_2^{(k)} + [(\varepsilon_j^{(k+1)})^4 + \\ & 4(\varepsilon_j^{(k+1)})^3 + 6(\varepsilon_j^{(k+1)})^2 + 4\varepsilon_j^{(k+1)}] \cdot \\ & X_{ij}^{(k)}(X_{ij}^{(k)})^3\overline{M}_3^{(k)}\end{aligned} \tag{11}$$

By solving simultaneously Eq. (5), Eq. (9), Eq. (11), the value of the $(k+1)$th $\varepsilon^{(k+1)}$ can be easily got without iteration analysis as Eq. (9) and Eq. (11) are explicit formulas.

Substituting Eqs. (9) and (11) into Eq. (5) and let

$$\Delta K_j^{(k+1)} = f_{K\varepsilon}(\varepsilon_j^{(k+1)})\Delta K(X_j^{(k)});$$
$$\Delta M_j^{(k+1)} = f_{M\varepsilon}(\varepsilon_j^{(k+1)})\Delta M(X_j^{(k)}).$$

gives the equation

$$\begin{aligned}\Delta\lambda_j^{(k+1)} = {} & (\varphi_j^{(k)})^T f_{K\varepsilon}(\varepsilon_j^{(k+1)})\Delta K(X_j^{(k)})\varphi_j^{(k)} - \\ & \lambda_j^{(k)}(\varphi_j^{(k)})^T f_{M\varepsilon}(\varepsilon_j^{(k+1)})\Delta M(X_j^{(k)})\varphi_j^{(k)}\end{aligned} \tag{12}$$

$$\varepsilon_j^{(k+1)} = f_\varepsilon\left(\frac{\Delta\lambda_j^{(k+1)}}{(\varphi_j^{(k)})^T\Delta K(X_j^{(k)})\varphi_j^{(k)} - \lambda_j^{(k)}(\varphi_j^{(k)})^T\Delta M(X_j^{(k)})\varphi_j^{(k)}}\right) \tag{13}$$

where $f_{K\varepsilon}$, $f_{M\varepsilon}$, f_ε are symbols of function. As

$$\Delta\lambda_j^{(k+1)} = \lambda_j^* - \lambda_j^{(k)},$$

$\Delta K(X_j^{(k)})$, $\Delta M(X_j^{(k)})$ can be solved from from Eqs. (9) and (11), and so $\varepsilon_j^{(k+1)}$ can be solved, too. And because $X^{(k)}$ is known, $X^{(k+1)} = (1+\varepsilon^{(k+1)})X^{(k)}$ can be solved.

For the sake of convenience, ignoring the high-order items of ε in Eqs. (9) and (11), the simpler equation can be given as follows

$$\begin{aligned}\Delta K_j^{(k+1)} = {} & \varepsilon_j^{(k+1)}X_{hj}^{(k)}\overline{K}_h^{(k)} - \varepsilon_j^{(k+1)}\frac{(X_{ij}^{(k)})^2\overline{K}_1^{(k)}}{(X_{ij}^{(k)})^3} + \\ & \frac{\varepsilon_j^{(k+1)}(X_{ij}^{(k)})^2\overline{K}_2^{(k)}}{X_{ij}^{(k)}} = \varepsilon_j^{(k+1)} \cdot \Delta\widetilde{K}_j^{(k)}\end{aligned} \tag{14}$$

$$\begin{aligned}\Delta M_j^{(k+1)} = {} & 2\varepsilon_j^{(k+1)}X_{ij}^{(k)}X_{ij}^{(k)}\overline{M}_1^{(k)} + \\ & 3\varepsilon_j^{(k+1)}X_{ij}^{(k)}(X_{ij}^{(k)})^2\overline{M}_2^{(k)} + \\ & 4\varepsilon_j^{(k+1)}X_{ij}^{(k)}(X_{ij}^{(k)})^3\overline{M}_3^{(k)}\varepsilon_j^{(k+1)}\Delta\widetilde{M}_j^{(k)}\end{aligned} \tag{15}$$

where $\Delta\widetilde{K}_j^{(k)}$, $\Delta M_j^{(k)}$ are the representatives of the kth iteration factor on the right of Eqs. (14) and (15).

$$\varepsilon_j^{(k+1)} = \frac{\Delta\lambda_j^{(K+1)}}{(\varphi_j^{(k)})^T\Delta\widetilde{K}_j^{(k)}\varphi_j^{(k)} - \lambda_j^{(k)}(\varphi_j^{(k)})^T\Delta\widetilde{M}_j^{(k)}\varphi_j^{(k)}} \tag{16}$$

Determination of the Inverse Parameter ε

Theoretically, ε can be obtained directly from Eq. (13) to meet Eq. (2). But if perturbed by the same value of ε, all of the design variables will scale and resize in the same proportion to $(1+\varepsilon)$, and of course this is unreasonable. However, Eq. (13) and Eq. (16) are still helpful and useful as they provide the theoretic basis of ε's evaluation. What should be pointed out is that the result computed through Eq. (13) or through simultaneous Eq. (5), (9) and (11) is only meet Eq. (5), a problem is too simple to have any constraint. While a structure system always has other constraints besides eigenvalue (natural frequency), i. e., stress range. Ref. 13 details this problem.

In fact, Eq. (16) is derived from the precondition of "all the design variables perturbed by a same perturbation parameter ε" where ε can be taken as the mean value of every design variable's corresponding perturbation parameters. Although it is unreasonable and unpractical to let distinguished types of design variables or different variables of same kind be perturbed with the same ε, sometimes the short-cut calculation of Eq. (16) is useful in practical application of engineering. Hence, it is possible to modify the computing result of Eq. (16) to meet the demands in feasible region and enabled conditions.

The ideal and feasible perturbation parameter for a inverse perturbation problem should not be a unique mean but be different values corresponding to the results of sensitivity analysis for design variables. Therefore if different correction factor can be selected for variant design variables according to the results of sensitivity analysis, the modified result $(\beta_i\varepsilon)$ is obviously reasonable and acceptable just like Eq. (16), here β_i is a modified coefficient.

1 The specified frequency

Suppose perturbation parameter $\varepsilon_i = \{\varepsilon_i \mid i = 1, 2, \cdots, n\}$ is corresponding to the perturbed design variables x_i. For convenience, omit the superscript k (iterations) and the subscript j (order of eigenvalue). Consider eigenvalue λ, from Eq. (4), let

$$\lambda'_i = \partial\lambda/\partial x_i,\ i = 1, 2, \cdots, n. \tag{17}$$

Define the mean absolute value $\lambda_{mean} = \sum_{i=1}^{n} 1/n \mid \lambda'_i \mid$, therefore the correction factor β_i is obtained

$$\beta_i = \frac{\mid \lambda'_i \mid}{\lambda_{mean}},\ i = 1, 2, \cdots, n \tag{18}$$

where $\beta_i > 0$, β_i is to resize the value of ε to get ε_i. So ε_i can be given by

$$\varepsilon_i = \beta_i\varepsilon,\ i = 1, 2, \cdots, n \tag{19}$$

where ε is obtained from Eq. (13) or (16).

2 Design with lower limit and upper limit on the variables

Consider the side constraints on the design variables and assume the lower limit and upper limit are X_{max}, X_{min} respectively, that is

$$X_{min} \leqslant X \leqslant X_{max} \tag{20}$$

From Eq. (8), $X_i^{(k+1)} = (1+\varepsilon_i)X_i^{(k)} = (1+\beta_i\varepsilon)X_i^{(k)}$, thus β_i should satisfy the following equation

$$\begin{cases} 0 < \beta_i \leqslant \dfrac{(X_{max} - X_i^{(k)})}{(X_i^{(k)} \cdot \varepsilon)} & (\text{when } \varepsilon > 0) \\ 0 < \beta_i \leqslant \dfrac{(X_{min} - X_i^{(k)})}{(X_i^{(k)} \cdot \varepsilon)} & (\text{when } \varepsilon < 0) \end{cases} \tag{21}$$

and β_i can be chosen as

$$\begin{cases} \beta_i = \min\left\{\beta_i, \dfrac{(X_{max} - X_i^{(k)})}{(X_i^{(k)} \cdot \varepsilon)}\right\} (\text{when } \varepsilon > 0) \\ \beta_i = \min\left\{\beta_i, \dfrac{(X_{min} - X_i^{(k)})}{(X_i^{(k)} \cdot \varepsilon)}\right\} (\text{when } \varepsilon < 0) \end{cases} \tag{22}$$

3 Design with frequency constraints bands

In fact, this inverse perturbation method can be used to solve either single frequency constraint band or multiple frequency constraint bands problem. Suppose the jth eigenvalue λ_j falls into one of the constraint band $\{\bar{\lambda}_m, \underline{\lambda}_m\}$, $\Delta\lambda_{jm}^{(k+1)}$ in Eq. (16) can be given by

$$\begin{cases} \Delta\bar{\lambda}_{jm}^{(k+1)} = \bar{\lambda}_m - \lambda_j^{(k)} \\ \Delta\underline{\lambda}_{jm}^{(k+1)} = \underline{\lambda}_m - \lambda_j^{(k)} \end{cases} \tag{22}$$

Once $\Delta\bar{\lambda}_{jm}^{(k+1)}$, $\Delta\underline{\lambda}_{jm}^{(k+1)}$ are obtained, the upper limit and lower limit $\bar{\varepsilon}_{jm}^{(k+1)}$, $\underline{\varepsilon}_{jm}^{(k+1)}$ of ε can be computed from Eq. (16). Assume that the number of constraint bands is p, namely, $m = 1, 2, \ldots, p$, thus

$$\begin{cases} \bar{\varepsilon} = \max\{\bar{\varepsilon}_{jm}^{(k+1)}\}, & (\bar{\varepsilon} > 0) \\ \underline{\varepsilon} = \min\{\underline{\varepsilon}_{jm}^{(k+1)}\}, & (\underline{\varepsilon} < 0) \end{cases}$$
$$m = 1, 2, \cdots, p;\ j = 1, 2, \cdots, n \tag{23}$$

If one hopes that the designed frequency upward perturbed to step aside the constraint band, the ε in Eq. (19) should be $\bar{\varepsilon}$; otherwise should be $\underline{\varepsilon}$. If only hopes to step aside the constraint band quickly and minimise the change of structure regardless of the perturbation direction, may be obtained from the following eqation

$$\varepsilon = \begin{cases} \bar{\varepsilon}, & \text{when } |\bar{\varepsilon}| < |\underline{\varepsilon}| \\ \underline{\varepsilon}, & \text{when } |\bar{\varepsilon}| > |\underline{\varepsilon}| \end{cases} \tag{24}$$

Algorithm Steps

The steps of the algorithm for the inverse perturbation method are as follows:

(1) Given an initial starting design variable, find the eigenvalues and eigenvectors from Eq. (2). Set the iteration index k to 0. If $|\lambda^* - \lambda^{(0)}|$ is sufficiently small, stop.

(2) Compute the $\varepsilon^{(k+1)}$ (If $\varepsilon^{(k+1)}$ is biggish, perturbed step by step) firstly from Eq. (16); Get perturbation coefficients $\varepsilon_i^{(k+1)}$ for each perturbed design variable by solving Eq. (19).

(3) Determine $\Delta K^{(k+1)}$ and $\Delta M^{(k+1)}$ by solving Eqs. (9) and (11). Then form new matrixes $K^{(k+1)}$ and $M^{(k+1)}$ using Eq. (8); compute the $\lambda^{(k+1)}$ form Eq. (2). If $|\lambda^* - \lambda^{(k+1)}|$ is sufficiently small, go to step 4, else set $k = k+1$ and repeat from step 2.

(4) Calculate $x_i^{(k+1)}$ and $K^{(k+1)}$ and $M^{(k+1)}$ again, using the matrixes to repeat analysis and repeat from step 1.

Numerical Examples

1 Example 1

Consider the simple system shown in Fig. 1[15]. The cantilever beam supporting the point mass $m_0 = \rho SL/10$ at its free end is composed of four prismatic segments of equal length L. Additional parameters are: $E = 1$, $S = 1$, $I = \pi d^4/64 = S^2/4\pi$.

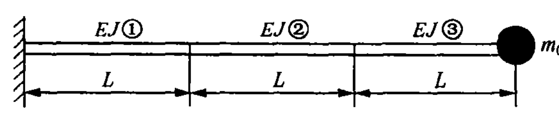

Fig. 1 Calculating structure model

where

L = element length

E = modulus of elasticity

S = cross-sectional area

ρ = density

d = cross-section diameter

There are only two degrees of freedom for each node because the axial rigidity $EF = \infty$. And the design variables are the element lengths of the three beam elements with initial values $X_L^{(0)} = \{0.96, 0.98, 0.97\}$. To get the specified fundamental natural frequency $\omega^* \geqslant 3.3023$, using the uniform mass matrix and obtained the results shown in Table 1. The corresponding optimal element lengths are $X_L^* = \{0.9957, 10138, 0.9941\}$, which is close to Ref. 15's results.

Table 1 Results for cantilever beam: optimum design variables for a specified first frequency

Iterative number	Target ω^*	Calculated ω_c	Perturbed elements	Perturbation coefficients ε_i	Perturbed design variables X_{Li} X_{L1}	X_{L2}	X_{L3}	Reanalyze (Y/N)?
0	3.3023	3.5118	/	/	0.96	0.98	0.97	Y
			1	0.0365				
1	3.3023	3.3061	2	0.0339	0.9950	1.0132	0.9937	N
			3	0.0244				

(Table 1 Continued)

Iterative Number	Target ω^*	Calculated ω_c	Perturbed elements	Perturbation Coefficients ϵ_i	Perturbed design variables X_{Li} X_{L1}	X_{L2}	X_{L3}	Reanalyze (Y/N)?
			1	$5.954*10^{-4}$				
2	3.3023	3.3028	2	$5.525*10^{-4}$	0.9956	1.0137	0.9941	N
			3	$3.985*10^{-4}$				
			1	$7.421*10^{-5}$				
3	3.3023	3.3024	2	$6.885*10^{-5}$	0.9957	1.0138	0.9941	N
			3	$4.966*10^{-5}$				
			1	$9.381*10^{-6}$				
4	3.3023	3.3023	2	$8.705*10^{-6}$	0.9957	1.0138	0.9941	N
			3	$6.278*10^{-6}$				
5	3.3023	3.3023	/	/	0.9957	1.0138	0.9941	Y

Table 2 Cantilever beam in flexural vibration: optimum design variables for a specified third eigenvalue

Iterative Number	Target λ_3^*	Calculated λ_{3c}	Perturbed elements	Perturbation coefficients ϵ_I	Perturbed design variables H_S(mm) H_{S1}	H_{S2}	H_{S3}	H_{S4}	H_{S5}	Reanalyze (Y/N)?
0	$2.109*10^7$	$2.291*10^7$	/	/	51.000	52.000	51.000	53.000	52.000	Y
			1	−0.0243						
			2	−0.0214						
1	$2.109*10^7$	$2.157*10^7$	3	−0.0085	49.759	50.828	50.568	50.767	51.847	N
			4	−0.0421						
			5	−0.0029						
			1	−0.0069						
			2	−0.0061						
2	$2.109*10^7$	$2.119*10^7$	3	−0.0021	49.415	50.576	50.461	50.134	51.816	N
			4	−0.0125						
			5	−0.0006						
			1	−0.0015						
			2	−0.0013						
3	$2.109*10^7$	$2.111*10^7$	3	−0.0004	49.341	50.509	50.438	49.997	51.810	N
			4	−0.0027						
			5	−0.0001						
			1	$-0.2956*10^3$						
			2	$-0.2631*10^3$						
4	$2.109*10^7$	$2.109*10^7$	3	$-0.0869*10^3$	49.327	50.496	50.434	49.970	51.809	N
			4	$-0.5408*10^3$						
			5	$-0.0220*10^3$						
5	$2.109*10^7$	$2.109*10^7$	/	/	49.327	50.496	50.434	49.970	51.809	Y

Table 3 Additive inverse problem: initial and final eigenvalues for a set of specified eigenvalues

Iterative number	λ_{c1}	λ_{c2}	λ_{c3}	λ_{c4}	λ_{c5}	λ_{c6}	λ_{c7}	λ_{c8}
λ_0	8.3108	19.9917	29.4978	40.0052	48.9947	58.2918	69.1453	85.7627
1	9.6754	20.1650	29.9690	39.9663	49.6444	58.7659	69.7271	81.6018
2	9.9213	20.0804	29.9953	39.9804	49.8557	59.2726	70.0778	80.6937
3	9.9773	20.0341	29.9980	39.9893	49.9281	59.5417	70.1809	80.1942
4	9.9930	20.0156	29.9991	39.9950	49.9638	59.7079	70.1890	80.1021
5	9.9978	20.0079	29.9996	39.9982	49.9831	59.8164	70.1598	80.0148
6	9.9993	20.0044	29.9998	39.9998	49.9933	59.8872	70.1207	79.9810
7	9.9997	20.0025	29.9999	40.0004	49.9982	59.9331	70.0846	79.9728
8	9.9999	20.0014	29.9999	40.0005	50.0003	59.9616	70.0560	79.9751
9	9.9999	20.0008	30.0000	40.0005	50.0009	59.9789	70.0353	79.9807
10	10.0000	20.0005	30.0000	40.0004	50.0010	59.9890	70.0214	79.9862

(Table 3 Continued)

Iterative number	λ_{c1}	λ_{c2}	λ_{c3}	λ_{c4}	λ_{c5}	λ_{c6}	λ_{c7}	λ_{c8}
11	10.000 0	20.000 2	30.000 0	40.000 3	50.000 8	59.994 7	70.012 4	79.990 8
12	10.000 0	20.000 1	30.000 0	40.000 2	50.000 6	59.997 7	70.006 9	79.994 1
13	10.000 0	20.000 1	30.000 0	40.000 1	50.000 4	59.999 3	70.003 6	79.996 4
14	10.000 0	20.000 0	30.000 0	40.000 1	50.000 3	60.000 0	70.001 7	79.997 8
15	10.000 0	20.000 0	30.000 0	40.000 0	50.000 2	60.000 2	70.000 7	79.998 8

2 Example 2

As is shown in Fig. 2, flexural vibration of a uniform[9], cantilever beam is used here to illustrate the method above. To simplify the problem, the beam cross section is rectangular, the motion will be planar, and the shear deformation and axial displacement will not be included. Consider the third natural frequency $\lambda_3 \geqslant 2.109\times10^7$ rad²/sec². $E = 2.068\,4\times10^5$ mPa, $\upsilon = 0.3$, $\rho = 7.833\,4\times10^{-9}$ Nsec²/mm⁴.

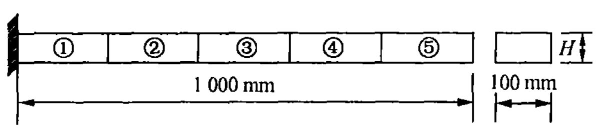

Fig. 2 Cantilever beam in flexural vibration

The initial design variables are $H_0 = \{51, 52, 51, 53, 52\}$, and the corresponding third natural frequency is $\lambda_3^{(0)} = 2.291 * 10^7$ rad²/sec². After 4 iteration got the results $H^* = \{49.33, 50.50, 50.43, 49.97, 51.81\}$, which is close to the Ref. 9's results. See Table 2.

3 Example 3

Given an additive inverse problem with distinct eigenvalue[16]. $A(c)$ be the family

$$A(c) = A_0 + \sum_{k=1}^{n} c_k A_k$$

Given real numbers $\lambda_1^* \leqslant \cdots \leqslant \lambda_n^*$, find $c \in \boldsymbol{R}^n$, such that $\lambda_i(c) = \lambda_i^*$, $i = 1, \cdots, n$. where $A(c)$ is real symmetric $n\times n$ matrices.

$$A_0 = \begin{bmatrix} 0 & & & & & & & \\ 4 & 0 & & & & & & \\ -1 & -1 & 0 & & & & & \\ 1 & 2 & 3 & 0 & & & & \\ 1 & 1 & 1 & 1 & 0 & & & \\ 5 & 4 & 3 & 2 & 1 & 0 & & \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{bmatrix},$$

$A_k = e_k e_k^T$, $k = 1, \cdots, 8$

$\lambda^* = (10, 20, 30, 40, 50, 60, 70, 80)$,

$c^0 = (10, 20, 30, 40, 50, 60, 70, 80)$.

Use the method above, the optimal eigenvalues can be gotten as $c^* = (11.907\,9, 19.705\,5, 30.545\,5, 40.062\,7, 51.587\,4, 64.703\,3, 70.170\,4, 71.317\,3)$. This is very similar to the exact solution $c^* = (11.970\,9, 19.705\,5, 30.545\,5, 40.062\,7, 51.587\,1, 64.702\,1, 70.170\,7, 71.318\,5)$ provided by Ref. 16. And it shows that the method prescribed in this paper is exact and effective. Table 3 displays the results.

4 Example 4

Although the method proposed in this paper is derived from the analysis of beam structure modeled with finite elements, it can be also applied to truss structure. Here a 25-bar truss problem that was provided by Ref. 6 is used to state the multiple frequency constraint bands problem. Fig. 3 shows the truss structure. All the elements are made of a material with Young's modulus $\rho = 7.0\times10^{11}$ N/m² and mass density $E = 2\,770$ kg/m³, gravity acceleration $g = 9.81$ m/s², nonstructural mass at all nodes $m_0 = 450$ kg and the minimum cross-sectional area is 0.5 cm². The target is to seek a design for which there are no frequencies in the bands 0.0 - 2.0 Hz and 14.0 - 21.0 Hz, which can be called as the first band and the second band, respectively.

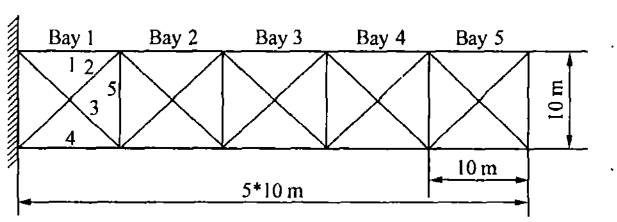

Fig. 3 Truss structure with 5 bays (25 bars)

Table 4 gives the results computed using our method in comparison with that given in Ref. 6. Table 5 gives the cross-sectional areas of the final design. Our results meet the design while Ref. 6's still fall in the prohibited second band.

Table 4 Calculated Frequencies Unit: Hz

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
Ref. 6	1.89	6.29	9.05	10.58	12.76	18.41	19.29
This paper	2.00	6.84	10.33	13.24	21.16	21.63	21.93

Table 5 Member Size of Final design for the truss Unit: cm²

Bays	Bar 1	Bar 2	Bar 3	Bar 4	Bar 5
1	61.43	20.05	20.05	54.04	5.00
2	46.91	20.05	17.52	37.81	11.21

(Table 5 Continued)

Bays	Bar 1	Bar 2	Bar 3	Bar 4	Bar 5
3	50.74	6.15	26.37	62.80	13.49
4	11.95	35.21	18.34	26.37	9.58
5	22.97	16.15	5.00	12.73	10.70

Summary

The perturbation parameter ϵ_i corresponding to design variable x_i can be calculated directly by substituting Eqs. (16) and (17) into Eq. (18). If the value of ϵ_i is not sufficiently small, and it can be also combined with the fractional step perturbation method to shorten the computing time, viz., replace $\Delta\lambda$ with $\Delta\lambda^c = \Delta\lambda / z$, where z is positive integer, then substitute $\Delta\lambda^c$ into Eq. (16). Moreover, the inverse perturbation method and structural matrix perturbation method can be used alternatively to solve this problem.

For a rotor system, the design of structure is not only with frequency (eigenvalue) constraint but also with other constraints such as stress constraint, etc. Therefore, the design optimization is actually a problem with multiple constraints. In the solving process the other constraints may be ignored temporally except the frequency constraint bands before designing meets the demands of optimization. Ref. 9 demonstrates this problem in detail.

The inverse perturbation method mentioned above can be also applied to truss structure and other structures or elements.

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